

# Logarithmic CFT on the Boundary and the World-Sheet

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## Abstract

The correspondences between logarithmic operators in the CFTs on the boundary of  $AdS_3$  and on the world-sheet and dipole fields in the bulk are studied using the free field formulation of the  $SL(2, C)/SU(2)$  WZNW model. We find that logarithmic operators on the boundary are related to operators on the world-sheet which are in indecomposable representations of  $SL(2)$ . The Knizhnik-Zamolodchikov equation is used to determine the conditions for those representations to appear in the operator product expansions of the model.

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# 1 Introduction

String theory on  $AdS_3$  is described by the  $SL(2, R)$  WZNW model (or its Euclidean version, the  $SL(2, C)/SU(2)$  coset model). This theory is interesting for a number of reasons. It provides a simple example of a string theory in a non-trivial background, and it is also one of the simplest examples of a non-rational CFT [1, 2]. It is closely related to string theory in black hole backgrounds such as the BTZ black hole [3] and the  $SL(2, R)/U(1)$  coset model which describes the two dimensional black hole [4]. There has been a great deal of interest in the model recently because it is also the simplest example of the  $AdS/CFT$  correspondence [5]–[12]. The model also has applications in condensed matter physics, to the theory of disordered systems [13] and the plateau transition in the quantum Hall effect [14].

In addition, the  $SL(2, R)$  WZNW model is thought to be an example of a Logarithmic CFT [15]. The first evidence of this was seen in [16], where an asymptotic solution for a correlation function of a gravitationally dressed CFT was found which had a logarithmic singularity, indicating that there are logarithmic operators in the theory. Two dimensional gravity has conserved currents which have the same Kac-Moody algebra as the currents in the  $SL(2, R)$  WZNW model [17], and the Ward identities which were used to find correlation functions in that model are equivalent to the Knizhnik-Zamolodchikov equations for correlation functions of the WZNW model. Thus, the logarithms that were found in some four-point functions in 2-dimensional gravity should also appear in the WZNW model [18]. Logarithmic behaviour has also been found in correlation functions of fields in finite dimensional representations of  $SL(2)$  in the  $SL(2, C)/SU(2)$  coset model [13], and recently in exact solutions of the Knizhnik-Zamolodchikov equations for correlation functions of fields in infinite dimensional representations of  $SL(2)$  as well [19, 20].

The presence of logarithmic operators in the spectrum of the  $SL(2, R)$  model raises several questions. The existence of a logarithmic operator always implies that the primary operator which is part of the same indecomposable representation of the Virasoro algebra has zero norm, so if all the ghosts are to be eliminated for the theory by the Virasoro constraints, we should not have any logarithmic operators. The no-ghost theorem for  $AdS_3$  restricts the fields that can appear in the string theory [21], and there may be a further restriction if the spectral flow is a symmetry of the theory [9]. It is therefore possible that when these restrictions are imposed, the fields which have logarithmic correlation functions are excluded, and there are no logarithmic operators. This is exactly the situation that occurs in the minimal models, or in WZNW models on compact groups – there are operators which have logarithmic correlation functions, but they are always outside the set of operators which occur in a unitary or minimal model. However, world-sheet logarithmic operators are thought to generate zero modes in target space which restore

symmetries that are broken by the string background [22]. This has mainly been used to study D-brane recoil (see eg. [23]), but logarithmic operators are expected to occur more generally in string theories in non-trivial backgrounds. One of the examples of a string theory with logarithmic operators discussed in [22] is the 2D black hole which is closely related to the  $SL(2, R)$  WZNW model. From this point of view, it would not be very surprising if string theory on  $AdS_3$  also has logarithmic operators.

There are two CFTs associated with the string theory on  $AdS_3$  - the world-sheet theory and the dual CFT on the boundary of  $AdS_3$  that is related to the string theory by the AdS/CFT correspondence. The dual theory can also be a LCFT, if there are singletons in the bulk of  $AdS_3$  [24, 25]. If singletons in  $AdS_3$  are related to logarithmic operators on the world sheet, we will therefore have a duality between two LCFTs.

In the next section we briefly review some basic facts about LCFT, the relation between singletons and logarithmic operators in the  $AdS_3$ /CFT correspondence, and the logarithms in correlation functions of the  $SL(2, R)$  WZNW model. In section (3) we use the free field formulation of the WZNW model to construct logarithmic operators, and determine how logarithmic operators on the world-sheet and on the boundary of  $AdS_3$  and fields in the bulk are related. In section (4) we use the Knizhnik-Zamolodchikov equations to investigate the conformal blocks and OPE of the WZNW model, and determine which OPEs include logarithmic operators.

## 2 Logarithmic CFT

A LCFT differs from an ordinary CFT in that the Virasoro generator  $L_0$  is not diagonalizable. In addition to the usual primary and descendant fields, it includes pairs of operators which form Jordan blocks for  $L_0$ :

$$L_0 C = hC, \quad L_0 D = hD + C \quad (1)$$

The fields  $C$  and  $D$  are therefore in a reducible, but indecomposable representation of the Virasoro algebra. This type of operator was first introduced in [15], and since then the theory of LCFTs has been developed in many papers, including [26]. The operators  $C$ ,  $D$  have the two-point functions

$$\begin{aligned} \langle C(z_1, \bar{z}_1) C(z_2, \bar{z}_2) \rangle &= 0 \\ \langle D(z_1, \bar{z}_1) C(z_2, \bar{z}_2) \rangle &= \frac{c}{|z_1 - z_2|^{4h}} \\ \langle D(z_1, \bar{z}_1) D(z_2, \bar{z}_2) \rangle &= \frac{1}{|z_1 - z_2|^{4h}} (d - 2c \ln |z_1 - z_2|^2) \end{aligned} \quad (2)$$

Here  $c$  is determined by the normalisation of  $C$  and  $D$  and  $d$  is arbitrary – it can be set to any value, using the symmetry of the theory under  $D \rightarrow D + \lambda C$ , which leaves (1)

unchanged. In the next section we will use the free field formulation of the WZNW model on  $SL(2, C)/(SU(2))$  to identify the logarithmic operators in the  $AdS_3/CFT$  correspondence, so we begin by reviewing the free field formulation of LCFTs with  $c < 1$ , which was developed in [27]. In that case, there is a single free field,  $\phi(z, \bar{z})$ , and the stress tensor is

$$T(z) = -\frac{1}{4}\partial_z\phi\partial_{\bar{z}}\phi + iQ\partial_z^2\phi \quad (3)$$

for a central charge  $c = 1 - 24Q^2$ . The primary fields are the exponentials of  $\phi$ :  $C_\alpha = e^{i\alpha\phi}$  with conformal weights  $h_\alpha = \bar{h}_\alpha = \alpha(\alpha - 2Q)$ . Logarithmic operators can be represented as derivatives of ordinary fields with respect to  $h$ :

$$D_\alpha = \frac{d}{dh}C_\alpha = \frac{dh}{d\alpha}\phi e^{i\alpha\phi} = \frac{i}{\alpha - Q}\phi e^{i\alpha\phi} \quad (4)$$

Then  $D_\alpha$  and  $C_\alpha$  form the Jordan block for  $L_0$  as in (1). However we cannot write down an operator of this form when  $\alpha = Q$ , so that  $h = -Q^2$  and  $\frac{dh}{d\alpha} = 0$ . In that case, the field  $\phi e^{i\alpha\phi}$  is a primary field – in Liouville theory it is the puncture operator. However, although there are no logarithms in the two point functions, there are logarithms in the four point functions of the field with this dimension. This indicates that logarithmic operators must appear in the operator product expansion (OPE) of the field with itself, and so the full spectrum of the theory must include logarithmic operators if it includes the primary operator with  $\alpha = Q$ . In an ordinary CFT the OPE of two primary fields has the form

$$O_1(z_1)O_2(z_2) = \sum_i \frac{f_{12}^i}{(z_1 - z_2)^{h_1+h_2-h-i}} O_i(z_2) + \dots \quad (5)$$

Where the  $(\dots)$  indicates the contributions of descendant fields. An OPE of this form implies an s-channel expansion of four point functions as

$$\langle O_i(z_1)O_j(z_2)O_k(z_3)O_l(z_4) \rangle = \sum_{kl} \frac{(f_{ij}^k)^2}{|z_{12}|^{h_i+h_j}|x_{34}|^{h_i+h_j}} F_{ij}^k(z)F_{ij}^k(\bar{z}) \quad (6)$$

where  $z = z_{12}z_{34}/z_{13}z_{24}$  and  $F_{ij}^k(z)$  has an expansion in powers of  $z$  around the point  $z = 0$ . In a LCFT, if one of the primary fields  $O_i$  is the field with dimension  $-Q^2/2$ , some of the conformal blocks instead have  $F_{ij}^k \sim z^{2h_k}(1 + \ln z + \dots)$ , which requires an OPE of the form

$$O_i(z_1)O_j(z_2) \sim \frac{1}{|z_1 - z_2|^{h_i+h_j-h}} (D + C \ln |z_1 - z_2|^2) + \dots \quad (7)$$

For that reason the field which has the form  $\phi e^{i\alpha\phi}$  but is not a logarithmic operator was called a pre-logarithmic operator in [27]. In the minimal models, with central charge  $c_{p,q} = 1 - 6(p-q)^2/pq$ , the field with dimension  $\alpha = Q$  is the degenerate primary field with dimension  $h_{p,q}$  at the corner of the Kac table. It is therefore excluded from the spectrum of the minimal models, which consists of operators with dimensions  $h_{r,s}$ ,  $1 \leq r \leq p-1$ ,

$1 \leq s \leq q - 1$ . However, it is possible to define expanded models with the same central charge as the minimal models, which include field with conformal weights from the edge of the Kac table, and these models do have logarithmic operators. To determine whether logarithmic operators can consistently be excluded from the spectrum of the  $SL(2, R)$  WZNW model, we therefore need to find out which primary fields will have logarithms in four point functions in that theory..

## 2.1 LCFT and Singletons

Since logarithmic operators are in indecomposable representations of the conformal algebra, in the AdS/CFT correspondence we can expect them to be related to fields which form similar indecomposable representations in the bulk of AdS. It was observed in [24] that singletons in AdS are objects which are in just that kind of representation [28]. A free singleton theory can be formulated in terms of a massive dipole field  $A$  which satisfies

$$(\nabla^2 - m^2)^2 A = 0 \quad (8)$$

When  $m^2 = -D^2/4$ , which is the lower bound for a stable massive scalar field on  $AdS_{D+1}$ , this equation has a singleton solution [29]. Eq. (8) can be solved by introducing a second field  $B$  with the equations of motion

$$\begin{aligned} (\nabla^2 - m^2)B &= 0 \\ (\nabla^2 - m^2)A - \mu^2 B &= 0 \end{aligned} \quad (9)$$

The action for field in  $AdS_{D+1}$  with these equations of motion is

$$S = \int d^{D+1}x \sqrt{g} \left( g^{\mu\nu} \partial_\mu A \partial_\nu B - m^2 AB - \frac{\mu^2}{2} B^2 \right) \quad (10)$$

Correlation functions of operators  $O_i(\vec{x})$  in the CFT on the boundary of  $AdS_{D+1}$  which correspond to fields  $\Phi_i$  in the bulk can be calculated from the bulk action using the relation [6, 7]

$$\left\langle \exp \left[ \sum_i \int d^D x \lambda_i(\vec{x}) O_i(\vec{x}_i) \right] \right\rangle_{CFT} = e^{-S[\{\Phi_i\}]|_{\Phi_i(\delta AdS) = \lambda_i(\vec{x})}} \quad (11)$$

In [25] and [24], this relation was used to find the two-point functions of operators on the boundary which correspond to the dipole fields with the action (10) in the bulk. The result is that for  $m^2 \neq -D^2/4$ , the two point functions are precisely those given by eq. (2), with  $h$  given by  $m^2 = 2h(2h - D)$ , indicating that dipole fields in the bulk lead to logarithmic operators with those dimensions on the boundary. In [24], but not in [25], it was found that the logarithmic correlation functions vanish for  $m^2 = -D^2/4$ ; we will return to this discrepancy in section (3.2).

## 2.2 Four-Point Functions in the WZNW model on $SL(2, R)$ or $SL(2, C)/SU(2)$

String theory on  $AdS_3$  is described by the WZNW model on  $SL(2, R)$ , or its Euclidean version  $SL(2, C)/SU(2)$ . This model has the  $\widehat{SL}(2) \times \widehat{SL}(2)$  algebra generated by left- and right-moving currents with the OPEs

$$\begin{aligned} J^3(z_1)J^3(z_2) &\sim -\frac{k}{2(z_1 - z_2)^2} \\ J^3(z_1)J^\pm(z_2) &\sim \pm \frac{J^\pm(z_2)}{z_1 - z_2} \\ J^-(z_1)J^+(z_2) &\sim \frac{k}{(z_1 - z_2)^2} + \frac{2J^3(z_2)}{z_1 - z_2} \end{aligned} \quad (12)$$

and similar relations for  $\bar{J}^a(\bar{z})$ . A primary operator  $\Phi_j(z, \bar{z})$  in the representation of  $SL(2)$  labeled by  $j$  is defined by the OPE

$$J^a(z_1)\Phi_j(z_2) \sim \frac{D^a\Phi_j}{z_1 - z_2} \quad (13)$$

where  $D^a$  are generators of  $SL(2)$ . It is useful to use the representation of the generators introduced in [30]

$$D^- = -\partial_x, \quad D^0 = x\partial_x + j, \quad D^+ = -x^2\partial_x - 2jx \quad (14)$$

so that the fields are now functions of two complex variables  $(x, \bar{x})$  and  $(z, \bar{z})$ . The stress tensor of the model is given by the Sugawara construction

$$T(z) = \frac{1}{k-2} : J^a(z)J^a(z) : = \frac{1}{k-2} : \left[ \frac{1}{2}J^+J^- + \frac{1}{2}J^-J^+ - J^0J^0 \right] : \quad (15)$$

So the primary fields have conformal dimensions  $h_j = \bar{h}_j = j(1-j)/(k-2)$ . The spectrum of the WZNW model includes operators in representations from both the principle continuous series of  $SL(2)$  with  $j = \frac{1}{2} + is$ , with  $s$  real, and the principle discrete representations with  $j$  real. There are also other representations which are obtained from these by the spectral flow [9], but we will not be interested in those representations in this paper. We will concentrate on operators with real  $j > 0$ , which correspond to local fields on the boundary of  $AdS_3$ . For normalizable functions on  $AdS_3$ , we should have  $j > 1/2$ , and for all negative-norm states to be removed by the no-ghost theorem, we need  $j < k/2$  [21]. If the spectral flow is a symmetry of theory, the spectrum is truncated further, and we have  $1/2 < j < (k-1)/2$  [9]. When  $j$  is equal to the upper or lower bound, a continuous representation appears ( $j = 1/2$  is part of the principle continuous series, and  $j - (k-1)/2$  is obtained from  $j = 1/2$  under the spectral flow). To determine

if the spectrum also includes logarithmic operators, the question we need to ask is therefore, are there logarithms in the four point functions (or OPEs) of primary fields with  $1/2 \leq j \leq (k-1)/2$ ?

Correlation functions of the WZNW model satisfy the Knizhnik-Zamolodchikov (KZ) equation

$$\left[ (k-2) \frac{\partial}{\partial z_i} + \sum_{j \neq i} \frac{D_i^a D_j^a}{z_i - z_j} \right] \langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle = 0 \quad (16)$$

For the WZNW on a compact group, the KZ equation reduces to a system of ordinary differential equations. For  $SL(2, R)$ , using the representations (14), we instead get a partial differential equation. The four point function can be expressed in terms of the cross ratios  $z = z_{12}z_{34}/z_{13}z_{24}$  and  $x = x_{12}x_{34}/x_{13}x_{24}$ , and  $z_{ij} = z_i - z_j$ , as

$$\begin{aligned} \langle \Phi_{j_1}(x_1, z_1) \cdots \Phi_{j_4}(x_4, z_4) \rangle &= |x_{24}^{-2j_2} x_{14}^{\beta_1} x_{13}^{\beta_2} x_{34}^{\beta_3} z_{24}^{-2h_2} x_{14}^{\gamma_1} z_{13}^{\gamma_2} z_{34}^{\gamma_3}|^2 G(x, \bar{x}, z, \bar{z}) \quad (17) \\ \beta_1 &= j_2 + j_3 - j_4 - j_1, \quad \beta_2 = j_4 - j_1 - j_2 - j_3, \quad \beta_3 = j_2 + j_1 - j_4 - j_3 \\ \gamma_1 &= h_2 + h_3 - h_4 - h_1, \quad \gamma_2 = h_4 - h_1 - h_2 - h_3, \quad \gamma_3 = h_2 + h_1 - h_4 - h_3 \end{aligned}$$

$G(x, \bar{x}, z, \bar{z})$  can be factorised into the conformal blocks, as

$$G(x, \bar{x}, z, \bar{z}) = \sum_{ij} U_{ij} F_i(x, z) F_i(\bar{x}, \bar{z}) \quad (18)$$

and the KZ equations for the conformal blocks are then [30, 11]

$$(k-2) \frac{\partial}{\partial z} F(x, z) = \left[ \frac{\mathcal{P}}{z} + \frac{\mathcal{Q}}{z-1} \right] F(x, z) \quad (19)$$

where

$$\begin{aligned} \mathcal{P} &= -x^2(1-x) \frac{\partial^2}{\partial x^2} + [(\tau+1)x^2 - 2j_1x - 2j_2x(1-x)] \frac{\partial}{\partial x} + 2j_2\tau x - 2j_1j_2 \\ \mathcal{Q} &= -x(1-x)^2 \frac{\partial^2}{\partial x^2} - [(\tau+1)(1-x)^2 - 2j_3(1-x) - 2j_2x(1-x)] \frac{\partial}{\partial x} \\ &\quad + 2j_2\tau(1-x) - 2j_3j_2 \end{aligned} \quad (20)$$

and  $\tau = j_1 + j_2 + j_3 - j_4$ . To find the complete four-point function, and the OPE of the operators in it, we need to determine which conformal blocks are included in a solution which satisfies the conditions of crossing symmetry and is single valued in the region of the singularities at  $x = 0, 1, \infty$ . The complete solutions for some four point functions, with logarithmic behaviour which requires logarithmic operators in the OPE, were found in [20, 19], but all the correlation functions calculated in which logarithms were found included operators with  $j < 1/2$ . For example if  $j_1 = j_3$  and  $j_2 = j_4 = 0$ , the solution is [20]

$$G(x, \bar{x}, z, \bar{z}) = \ln \left| \frac{1-x}{x} \right| + \frac{2j_1-1}{k-2} \ln \left| \frac{1-z}{z} \right| \quad (21)$$

This correlation function involves the operator in a non-trivial representation with  $j = 0$ , which is not expected to be part of the spectrum of string theory on  $AdS_3$ , although it may be relevant to the theory for the Quantum Hall plateau transition [14, 19]. However, the KZ equation has a symmetry which can be used to relate this four point function to the four point function with  $j_2 = j_4 = 1$ . If  $1 - 2j_2$  is a non-negative integer, then  $\frac{\partial^{1-2j_2}}{\partial x^{1-2j_2}} F(x, z)$  is a solution to the KZ equation with  $j_2$  replaced by  $1 - j_2$  [20]. Using this symmetry we can see that the four point function with  $j_2 = 1$ ,  $j_1 = j_3$ ,  $j_4 = 0$  has no logarithms – it is simply

$$G(x, \bar{x}, z, \bar{z}) = -\frac{1}{x} - \frac{1}{1-x} - \frac{1}{\bar{x}} - \frac{1}{1-\bar{x}} \quad (22)$$

Therefore the correlation functions calculated in [19, 20] do not prove that there are logarithmic operators in the WZNW model. We will look more closely at correlation function of operators with  $j \geq 1/2$  in section (4).

### 3 Logarithmic Operators in $AdS_3$

#### 3.1 Semiclassical Approach

The action for the WZNW model on  $SL(2, C)/SU(2)$  can be written using the Gauss parameterization of  $SL(2, C)$  as [2]

$$S = k \int d^2 z \left[ \partial \phi \bar{\partial} \phi + \bar{\partial} \gamma \partial \bar{\gamma} e^{2\phi} \right] \quad (23)$$

This action describes strings propagating in the Euclidean  $AdS_3$  target space, with  $(\phi, \gamma, \bar{\gamma})$  being coordinates on  $AdS_3$ . The zero modes of the currents are realised by

$$J_0^- = \partial_\gamma, \quad J_0^0 = \gamma \partial_\gamma - \frac{1}{2} \partial_\phi, \quad J_0^+ = \gamma^2 \partial_\gamma - \gamma \partial_\phi - e^{-2\phi} \partial_{\bar{\gamma}} \quad (24)$$

with similar expressions for  $\bar{J}^a$ . Primary fields  $\Phi_j(z)$  satisfy

$$J_0^a \Phi_j = D^a \Phi_j, \quad J_n^a \Phi_j = 0, \quad n > 0 \quad (25)$$

with  $D^a$  given by (14). The primary fields which satisfy (25) are [11]

$$\Phi_j = \left[ \frac{1}{(\gamma - x)(\bar{\gamma} - \bar{x})e^\phi + e^{-\phi}} \right]^{2j} \quad (26)$$

These fields can be expanded as

$$\Phi_j(x, \bar{x}, z, \bar{z}) = \sum_m x^{-j+m} \bar{x}^{-j+\bar{m}} \Phi_j^m(z, \bar{z}) \quad (27)$$



The vertex operator (26) has the form of a boundary-bulk Green function for a scalar field on  $AdS_3$ , with mass given by

$$m^2 = 4j(1-j) \quad (28)$$

This is because  $\Phi_j$  is an eigenfunction the Laplacian on  $AdS_3$ , which in the  $(\phi, \gamma, \bar{\gamma})$  coordinates is just  $\nabla^2 = 2(J_0^a J_0^a + \bar{J}_0^a \bar{J}_0^a)$ , with the  $J_0^a$  given by eq. (24). The new variables  $x, \bar{x}$  can therefore be considered as coordinates on the boundary of  $AdS_3$  [8]

As in the  $c_{p,q}$  models, we can expect logarithmic fields to be given by the derivative with respect to  $h_j$  of the primary fields. We therefore define

$$\tilde{\Phi}_j = \frac{d}{dh_j} \Phi_j = \left[ \frac{1}{(\gamma - x)(\bar{\gamma} - \bar{x})e^\phi + e^{-\phi}} \right]^{2j} \ln \left[ (\gamma - x)(\bar{\gamma} - \bar{x})e^\phi + e^{-\phi} \right]^2 \quad (29)$$

so that  $\Phi_j$  and  $\tilde{\Phi}_j$  satisfy

$$L_0 \Phi_j = h_j \Phi_j, \quad L_0 \tilde{\Phi}_j = h_j \tilde{\Phi}_j + \frac{1-2j}{k-2} \Phi_j \quad (30)$$

This has the same form of (1), with  $C = \Phi_j$  and  $D = \frac{k-2}{1-2j} \tilde{\Phi}_j$  giving the conventional normalization of the logarithmic pair. We can also see that when  $j = 1/2$ ,  $L_0$  becomes diagonal and so  $\tilde{\Phi}_{\frac{1}{2}}$  is not a logarithmic operator. The action of the currents on  $\tilde{\Phi}_j$  is

$$\begin{aligned} J_0^- \tilde{\Phi}_j &= D^- \tilde{\Phi}_j \\ J_0^0 \tilde{\Phi}_j &= D^0 \tilde{\Phi}_j + \Phi_j \\ J_0^+ \tilde{\Phi}_j &= D^+ \tilde{\Phi}_j + 2x \Phi_j \end{aligned} \quad (31)$$

where  $D^0, D^\pm$  are given by eq. (14); also  $J_n^a \tilde{\Phi}_j = 0$  for  $n \geq 1$ . The operators  $\tilde{\Phi}_j$  and  $\Phi_j$  are therefore in an indecomposable representation  $SL(2)$  as well as of the Virasoro algebra. However, these representations are not the same as the indecomposable representations of the Kac-Moody algebra which were discussed in [18, 27]. The difference is that in those representations, the Casimir operator  $J_0^a J_0^a$  was diagonal, and so the operators in those representations were not necessarily logarithmic operators as far as the Virasoro algebra was concerned. However, we will see that the type of four point functions which were analyzed in [18] are consistent with OPEs which include the  $\tilde{\Phi}_j$  type of field. Eq. (31) also applies when  $j = 1/2$ , so that  $\tilde{\Phi}_{\frac{1}{2}}$  is in an indecomposable representation of  $SL(2)$  even though it is an ordinary primary field in an irreducible representation of the Virasoro algebra.

Eq. (31) is consistent with eq. (30), since  $J_0^a J_0^a \tilde{\Phi}_j = j(1-j) \tilde{\Phi}_j + (1-2j) \Phi_j$ . Because  $J_0^a J_0^a$  is the Laplacian on  $AdS_3$ , this means that  $\tilde{\Phi}_j$  is the boundary-bulk Greens function for dipole fields with the equation of motion (9), just as the usual vertex operators  $\Phi_j$

are boundary-bulk Greens functions for scalar fields. As before the mass is given by (28). In fact this follows immediately from the way we defined  $\tilde{\Phi}_j$ , since in [25, 24] this Green function was already found to be given by differentiating the Green function for scalar fields with respect to  $m^2$  or  $j$ . We can therefore identify dipole fields in the bulk with vertex operators on the world sheet which are in indecomposable representations of the global  $SL(2)$  symmetry. To see directly that these vertex operators lead to logarithmic operators in the CFT on the boundary, we need to show that the correlation functions of these operators have logarithmic dependence on  $x$  as well as  $z$ .

We can use eq. (31) to derive Ward identities for correlation functions of the fields  $\tilde{\Phi}_j$ . These are just the same as the Ward identities for the logarithmic pair of operators in an ordinary LCFT, with  $z$  replaced by  $x$  and  $h$  replaced by  $j$ :

$$\begin{aligned}
[D_{(1)}^a + D_{(2)}^a] \langle \Phi_j(x_1, z_1) \Phi_j(x_2, z_2) \rangle &= 0, \quad a = \pm, 0 \\
[D_{(1)}^- + D_{(2)}^-] \langle \tilde{\Phi}_j(x_1, z_1) \Phi_j(x_2, z_2) \rangle &= [D_{(1)}^- + D_{(2)}^-] \langle \tilde{\Phi}_j(x_1, z_1) \tilde{\Phi}_j(x_2, z_2) \rangle = 0 \\
[D_{(1)}^0 + D_{(2)}^0] \langle \tilde{\Phi}_j(x_1, z_1) \Phi_j(x_2, z_2) \rangle &= -\langle \Phi_j(x_1, z_1) \Phi_j(x_2, z_2) \rangle \\
[D_{(1)}^+ + D_{(2)}^+] \langle \tilde{\Phi}_j(x_1, z_1) \Phi_j(x_2, z_2) \rangle &= -2x_1 \langle \Phi_j(x_1, z_1) \Phi_j(x_2, z_2) \rangle \\
[D_{(1)}^0 + D_{(2)}^0] \langle \tilde{\Phi}_j(x_1, z_1) \tilde{\Phi}_j(x_2, z_2) \rangle &= -\langle \tilde{\Phi}_j(x_1, z_1) \Phi_j(x_2, z_2) \rangle - \langle \Phi_j(x_1, z_1) \tilde{\Phi}_j(x_2, z_2) \rangle \\
[D_{(1)}^+ + D_{(2)}^+] \langle \tilde{\Phi}_j(x_1, z_1) \tilde{\Phi}_j(x_2, z_2) \rangle &= -2x_1 \langle \Phi_j(x_1, z_1) \tilde{\Phi}_j(x_2, z_2) \rangle - 2x_2 \langle \tilde{\Phi}_j(x_1, z_1) \Phi_j(x_2, z_2) \rangle
\end{aligned} \tag{32}$$

Where  $D_{(x_i)}^a$  is given by (14) with  $x = x_i$ , and there are similar equations for the  $\bar{x}$  dependence. The conformal Ward identities lead to two point functions of the form (2) for  $C = \Phi_j$  and  $D = \frac{k-2}{1-2j} \tilde{\Phi}_j$ , except that  $c$  and  $d$  in (2) can now depend on  $x$ . The solution to eq. (32) for the two point functions are then

$$\begin{aligned}
\langle \Phi_j(x_1, z_1) \Phi_j(x_2, z_2) \rangle &= 0 \\
\langle \tilde{\Phi}_j(x_1, z_1) \Phi_j(x_2, z_2) \rangle &= \frac{c'}{|z_1 - z_2|^{4h_j} |x_1 - x_2|^{4j}} \\
\langle \tilde{\Phi}_j(x_1, z_1) \tilde{\Phi}_j(x_2, z_2) \rangle &= \frac{c'}{|z_1 - z_2|^{4h_j} |x_1 - x_2|^{4j}} \left[ d' - 2 \frac{1-2j}{k-2} \ln |z_1 - z_2|^2 - 2 \ln |x_1 - x_2|^2 \right]
\end{aligned} \tag{33}$$

since  $(x, \bar{x})$  are the coordinates on the boundary of  $AdS_3$ , this confirms that these fields also give logarithms in the CFT on the boundary. Of course, to find the complete correlation function in the string theory we would have to include the contribution from the internal CFT, and integrate over  $z_i$ , but that cannot change the way the correlation functions depend on  $x$ . The correspondence we have is therefore

$$\tilde{\Phi}_j(\text{world-sheet CFT}) \longrightarrow \text{dipole fields in } AdS_3 \longrightarrow \text{LCFT on boundary of } AdS_3$$

### 3.2 $j = \frac{1}{2}$

When  $j = 1/2$ , the  $\ln z$  term in the two point function vanishes, and  $L_0$  becomes diagonalizable in eq. (30). Thus  $\tilde{\Phi}_{\frac{1}{2}}$ , like the puncture operator in Liouville theory, has the form of a logarithmic operator but is actually a primary operator. However, it is still in an indecomposable representation of the Kac-Moody algebra, and thus we can expect that other indecomposable representations will appear in the OPEs of  $\tilde{\Phi}_{\frac{1}{2}}$  with other primary fields. Since operators in the other indecomposable representations are logarithmic operators, this means that if  $\tilde{\Phi}_{\frac{1}{2}}$  is part of the spectrum of the WZNW model, there will also be logarithmic operators in the spectrum. This is the same situation as is familiar for the pre-logarithmic puncture-type operators in the  $c_{p,q}$  LCFTs, except that in this case, we can see that it must occur because the pre-logarithmic  $\tilde{\Phi}_{\frac{1}{2}}$  operator is already in an indecomposable representation of the full symmetry algebra of the model.

The  $\ln x$  term in eq. (34) does not vanish for  $j = 1/2$ , and so it will still be a logarithmic operator of the CFT on the boundary. This appears to contradict the result of [24], where it was found that for the singleton with mass  $m^2 = -1$ , which corresponds to the vertex operators with  $j = 1/2$ , all the logarithmic correlation functions on the boundary become null. However, we can check that there is no real contradiction, because the dipole fields with  $m^2 = -1$  considered in [24] and [25] actually have different actions and equations of motion, so provided  $\tilde{\Phi}_j$  turns out to be the boundary-bulk Green function for the action considered in [25], it will indeed couple to a logarithmic operator on the boundary.

In both [25] and [24], the action has the form of eq. (10), but in [24]  $\mu$  is taken to be a constant,  $\mu^2 = 1$ , while in [25]  $\mu$  is taken to be given by  $\mu^2 = 4j - 2$  (in 3 dimensions). When  $\mu \neq 0$  the two actions are equivalent, since  $\mu^2$  can always be set equal to 1 by replacing  $A$  and  $B$  with  $A' = \mu A$  and  $B' = \mu^{-1} B$ . When  $j = 1/2$  the two versions of the action are inequivalent, and if  $\mu \neq 0$  the result of [24] that there are no logarithms on the boundary will apply. It is therefore more useful for us to take  $\mu^2 = 4j - 2$  as in [25], so that when  $j = 1/2$ ,  $\mu = 0$ , and the action (10) reduces to

$$S = \int d^3x \sqrt{g} (g^{\mu\nu} \partial_\mu A \partial_\nu B + AB) \quad (34)$$

and the equations of motion are simply

$$(\nabla^2 + 1)B = (\nabla^2 + 1)A = 0 \quad (35)$$

It was observed in [31] that in the  $AdS_3$ /CFT correspondence, a logarithmic operator on the boundary with dimension 1, corresponding to  $m^2 = -1$ , would have equations of motion in the bulk with no diagonal terms, but that was interpreted as meaning that there could be no logarithmic operators on the boundary with  $m^2 = -1$ . To see why this is not necessarily true, we can see what happens when we start from the equations which

determine the Green functions for  $j \neq 1/2$ , and take  $j \rightarrow 1/2$ . The Green functions  $K_{ij}$  are determined by the equations

$$\begin{aligned} (\nabla^2 - m^2) K_{AA} - \mu^2 K_{BA} &= 0, & (\nabla^2 - m^2) K_{BB} &= 0 \\ (\nabla^2 - m^2) K_{AB} - \mu^2 K_{BB} &= 0, & (\nabla^2 - m^2) K_{BA} &= 0 \end{aligned} \quad (36)$$

together with the boundary condition that  $K_{ij} = 0$  on the boundary of  $AdS_3$ . Even when  $j \neq 1/2$ , these equations have more than one solution, depending on whether we take  $K_{ij}$  to be symmetric or not [24]. If  $K_{ij}$  is symmetric, the solution is

$$K_{BB} = 0, \quad K_{AB} = K_{BA} = K, \quad K_{AA} = \frac{1}{2} \frac{dK}{dj} \quad (37)$$

where  $K$  is the boundary-bulk Green function for a massive scalar field, which was found in [7], and in the  $(\phi, \gamma, \bar{\gamma}; x, \bar{x})$  coordinates is  $\Phi_j$ . The other independent solution is

$$K_{BA} = 0, \quad K_{AA} = K_{BB} = K, \quad K_{AB} = \frac{1}{2} \frac{dK}{dj} \quad (38)$$

Although there are two choices for the Green functions they both lead to the same two point functions on the boundary. The only differences is that in the symmetric case, the field  $A$  couples to the logarithmic operator  $D$  on the boundary and  $B$  couples to the primary operator  $C$ , while in the non-symmetric case,  $A$  couples to  $C$  and  $B$  to  $D$  [24]. If we choose the normalization of [25], both solutions still apply when  $j = 1/2$ , because then  $\frac{dm^2}{dj} = 0$  and so  $\frac{d}{dj}$  commutes with  $(\nabla^2 - m^2)$ . The calculation of the two point functions using (11) then proceeds in exactly the same way when  $j = 1/2$  as when  $j \neq 1/2$ , so either choice still leads to logarithms on the boundary. The difference is that there is now a third solution for the Green functions, which is just  $K_{AA} = K_{BB} = K$ ,  $K_{AB} = K_{BA} = 0$ . Since the Green functions are then the same as for ordinary scalar fields, there will be no logarithms on the boundary if we make this choice. We therefore cannot determine if there will be logarithmic operators on the boundary just using the action (34) – we need some information about interactions in the theory. Instead, we can try to determine whether the spectrum of the CFT on the world-sheet includes  $\tilde{\Phi}_{\frac{1}{2}}$ , which would indicate that we do have a logarithmic operator on the boundary, or only  $\Phi_{\frac{1}{2}}$  in which case there are no logarithmic operators.

## 4 Conformal Blocks and OPEs

To decide if there will be logarithmic operators in the theory, we need to know if only the operator  $\Phi_{\frac{1}{2}}$  occurs, or if  $\tilde{\Phi}_{\frac{1}{2}}$  appears as well. We are therefore interested in the OPEs

of primary fields for which  $\Phi_{\frac{1}{2}}$ , and possibly also  $\tilde{\Phi}_{\frac{1}{2}}$  occur. If there are no logarithmic operators, the leading terms in the OPEs are

$$\Phi_{j_1}(x_1, z_1)\Phi_{j_2}(x_2, z_2) \sim \sum_{j_3} \frac{C(j_1, j_2, j_3)}{|x_{12}|^{2(j_1+j_2-j_3)}|z_{12}|^{2(h_1+h_2-h_3)}}\Phi_{j_3}(x_2, z_2) \quad (39)$$

In general there is also another contribution to the OPE which is an integral over  $j = 1/2 + is$ , which we ignore here. If all the OPEs are of this form, there can be no logarithms in correlation functions. The other possibility is that, when  $j_3 = 1/2$  appears in the OPE, it becomes

$$\begin{aligned} \Phi_{j_1}(x_1, z_1)\Phi_{j_2}(x_2, z_2) &\sim \frac{C(j_1, j_2, \frac{1}{2})}{|x_{12}|^{2(j_1+j_2-j_3)}|z_{12}|^{2(h_1+h_2-h_3)}} \left[ \ln |x_{12}|^2 \Phi_{\frac{1}{2}}(x_2, z_2) + \tilde{\Phi}_{\frac{1}{2}}(x_2, z_2) \right] \\ &+ \sum_{j_3} \frac{C(j_1, j_2, j_3)}{|x_{12}|^{2(j_1+j_2-j_3)}|z_{12}|^{2(h_1+h_2-h_3)}}\Phi_{j_3}(x_2, z_2) \end{aligned} \quad (40)$$

In either case the OPE also has descendant terms of higher order in  $x$  and  $z$ . The possible values of  $j_3$  which occur in the OPE can be determined from the three point functions of primary fields which were calculated in [11, 12]. They are give by the poles of the function (from eq. (5.29) of [12])

$$D(j_a) \sim \left( \frac{1}{k} b^{-2b^2} \frac{\Gamma(b^2)}{\Gamma(1-b^2)} \right)^{1-j_1-j_2-j_3} \frac{1}{\Upsilon((j_1+j_2+j_3-1)b)} \sum_{a=1}^3 \frac{\Upsilon(2j_a b)}{\Upsilon((j_1+j_2+j_3-2j_a)b)} \quad (41)$$

where  $b^2 = 1/(k-2)$  and  $\Upsilon(x)$  is the  $\Upsilon$ -functions defined in [32].  $\Upsilon(x)$  has zeros at

$$x = -mb - \frac{n}{b}, \quad (m+1)b + \frac{n+1}{b}, \quad m, n \in \mathbf{Z}_{\geq 0} \quad (42)$$

and so  $D(j_1, j_2, j_3)$  has poles when one of the combinations  $j_1+j_2-j_3$ ,  $j_2+j_3-j_1$ ,  $j_3+j_1-j_2$  or  $j_1+j_2+j_3-1$  takes one of the values  $(m+1) + (n+1)(k-2)$  or  $-m - n(k-2)$ , for a pair of non-negative integers  $(m, n)$ . To have  $j_3 = 1/2$ , we can therefore choose  $j_1, j_2$  so that  $2(j_1 - j_2) \in \mathbf{Z}$ . However, we cannot tell from the three-point function which type of OPE we have, because both (39) and (40) lead to the same three point function for  $\langle \Phi_{j_1} \Phi_{j_2} \Phi_{\frac{1}{2}} \rangle$ . The logarithmic term in eq. (40) does not contribute because the two point function of  $\Phi_{\frac{1}{2}}$  must be null, as in eq. (34), if  $\tilde{\Phi}_{\frac{1}{2}}$  is in the OPE. To decide which OPE is correct, we can consider the four point functions

$$\langle \Phi_{j_1}(x_1, z_1)\Phi_{j_2}(x_2, z_2)\Phi_{j_1}(x_3, z_3)\Phi_{j_2}(x_4, z_4) \rangle \quad (43)$$

which can be written in the form of eqs. (17,18) with  $j_3 = j_1$  and  $j_4 = j_2$ . Whichever OPE is correct (provided there are no  $\ln z$  terms), the s-channel conformal block which contains the contribution of  $\Phi_J$  to the OPE of  $\Phi_{j_1}$  and  $\Phi_{j_2}$  can be written as

$$F_J(x, z) = z^{h_J-h_1-h_2} \sum_{n=0}^{\infty} z^n F_J^n(x) \quad (44)$$

The function  $F_J^n(x)$  then contains all the contributions to the OPE of  $\Phi_{j_1}\Phi_{j_2}$  which are descendants of  $\Phi_J$  of Virasoro level  $n$ . In particular,  $F_J^0(x)$  determines the OPE coefficients of all the operators  $(J_0^-)^n\Phi_J$ . The KZ equation (19) becomes a system of equations for  $F_J^n(x)$ :

$$[\mathcal{P} - (k-2)(h_J - h_1 - h_2 - n)] F_J^n = [\mathcal{P} + \mathcal{Q} - (k-2)(h_J - h_1 - h_2 - n + 1)] F_J^{n-1} \quad (45)$$

where  $\mathcal{P}$  and  $\mathcal{Q}$  are given by eq. (20) with  $j_3 = j_1$  and  $j_4 = j_2$ . If we now write  $F_J^0(x) = x^{J-j_+}\tilde{F}_J^0$ ,  $j_\pm = j_1 \pm j_2$ , the equation for  $\tilde{F}_J^0$  becomes a hypergeometric equation and the solution is

$$F_J^0(x) = f_J(x) \equiv x^{J-j_+} F(J+j_-, J-j_-; 2J; x) \quad (46)$$

The other solution of the equation for  $F_J^0$  is  $f_{1-J}(x)$ , provided  $2J$  is not an integer, so in order to have an OPE which only includes operators  $\Phi_J$  with  $J \geq 1/2$ , we have to choose the first solution. If  $J \geq 1$  and  $2J \in \mathbf{Z}$ , the second solution instead has the form

$$f_J^{(2)}(x) = \ln x f_J(x) + x^{1-J-j_+} H(x) \quad (47)$$

where  $H(x)$  is a function which is regular at  $x = 0$ . A four point function of this form would imply both that the OPE included a primary field with spin  $1 - J$  which is less than  $1/2$ , and that the OPE has a  $\ln x$  term but no  $\ln z$  term, which from the previous section we know can only happen for  $J = 1/2$ , so we again have to take the first solution. If  $j_1 = j_2$ , the identity operator with  $J = 0$  contributes to the s-channel expansion and we have

$$f_0(x) = x^{-2j_1} F(0, 0; 0; x) = x^{-2j_1} \quad (48)$$

In this case  $F_{J=0}^0$  has only a single term, since the operator  $\Phi_0$  in the OPE is just the identity, and not the non-trivial operator with  $J = 0$  we considered in section (2), so  $J_0^a\Phi_0 = 0$  and does not appear in the OPE. The second solution in this case is  $f_1(x)$ , which gives the conformal block for  $J = 1$ . In all cases, Eq. (45) can be turned into a set of algebraic recursion relations using

$$\begin{aligned} \mathcal{P}f_J(x) &= [(J(1-J) - j_1(1-j_1) - j_2(1-j_2))] f_J(x) \\ [\mathcal{P} + \mathcal{Q}] f_J(x) &= \left[ -x(1-x)\frac{d^2}{dx^2} + (2x-1)\frac{d}{dx} \right] f_J(x) \\ &= \sum_{J'=J, J\pm 1} C_J^{J'} f_{J'}(x) \end{aligned} \quad (49)$$

where

$$C_J^{J-1} = -(J-j_+)^2$$

$$\begin{aligned}
C_J^J &= -\frac{1}{2} \left[ J(J-1)^2 - j_+(j_+-2) - j_-^2 + \frac{(j_+-1)^2 j_-^2}{J(J-1)} \right] \\
C_J^{J+1} &= \left[ \frac{(j_-+J)(j_--J)}{4J^2(2J+1)(2J-1)} \right] \\
&\quad \times \left[ J^2((J-1)^2 + 2j_+(J-1) + j_+^2 - j_-^2) + 2Jj_-^2(1-j_+) - j_-^2(j_+-1)^2 \right]
\end{aligned} \tag{50}$$

Thus,  $F_J^n(x)$  can be expressed as a sum of the functions  $f_K(x)$ , with  $K = J, J \pm 1, \dots, J \pm n$ , for any  $J \neq 1/2$ , and, as we expected, the OPE has the form of eq. (39) with no logarithmic operators. The one remaining case to consider is  $J = 1/2$ . This can occur if  $2j_- \in \mathbf{Z}$ , so the simplest possibility is to take  $j_- = 1/2$ . Then we have

$$f_{\frac{1}{2}} = x^{\frac{1}{2}-j_+} F(1, 0; 1; x) = x^{\frac{1}{2}-j_+} \tag{51}$$

This will give us a conformal block with no logarithms, but a solution of this form implies that  $\Phi_{\frac{1}{2}}$  appears in the OPE but  $J_0^- \Phi_{\frac{1}{2}}$  does not, so that the three point functions satisfy

$$\begin{aligned}
\langle \Phi_{j_1}(x_1, z_1) \Phi_{j_2}(x_2, z_2) \Phi_{\frac{1}{2}}(x_3, z_3) \rangle &\neq 0 \\
\langle \Phi_{j_1}(x_1, z_1) \Phi_{j_2}(x_2, z_2) [J_0^- \Phi_{\frac{1}{2}}(x_3, z_3)] \rangle &= -\frac{\partial}{\partial x_3} \langle \Phi_{j_1}(x_1, z_1) \Phi_{j_2}(x_2, z_2) \Phi_{\frac{1}{2}}(x_3, z_3) \rangle = 0
\end{aligned} \tag{52}$$

These cannot both be true, so the only way to get an OPE with no  $\ln x$  terms is if  $\Phi_{\frac{1}{2}}$  does not appear at all. To get an acceptable OPE including  $\Phi_{\frac{1}{2}}$  we therefore have to take the general solution for  $F_{\frac{1}{2}}^0$ , which is

$$F_{\frac{1}{2}}^0(x) = x^{\frac{1}{2}-j_+} \left[ A \ln \left( \frac{x}{1-x} \right) + B \right] \tag{53}$$

and the recursion relations for  $F_{\frac{1}{2}}^n$  are then

$$\left[ \mathcal{P} - \frac{1}{8} + \frac{j_+^2}{2} - j_+ + n(k-2) \right] F_{\frac{1}{2}}^n = \left[ \mathcal{P} + \mathcal{Q} - \frac{1}{8} + \frac{j_+^2}{2} - j_+ + (n-1)(k-2) \right] F_{\frac{1}{2}}^{n-1} \tag{54}$$

A conformal block of the form (53) implies that the leading term in the OPE must have the form of eq. (40), and so we can see that it is impossible to have  $\Phi_{\frac{1}{2}}$  in the spectrum of the WZNW model without  $\tilde{\Phi}_{\frac{1}{2}}$  also being included.

Finally, we can show that the inclusion of  $\tilde{\Phi}_{\frac{1}{2}}$  also implies that the spectrum must include other logarithmic operators, which will have correlation functions which depend on  $\ln z$  as well as  $\ln x$ , by considering the OPE of  $\tilde{\Phi}_{\frac{1}{2}}$  and  $\Phi_j$ . The form of the OPE for  $\Phi_{\frac{1}{2}} \Phi_j$  will as usual be given by eq. (39), and then the OPE of  $\tilde{\Phi}_{\frac{1}{2}} \Phi_j$  can be found by applying the currents  $J_0^a$  to both sides of the OPE, or more simply by differentiating eq. (39) wrt  $j_1$ , giving

$$\tilde{\Phi}_{\frac{1}{2}}(x_1) \Phi_j(x_2) = \sum_{j_3} \frac{C(\frac{1}{2}, j, j_3)}{|x_{12}|^{2(1+j-j_3)} |z_{12}|^{2(h_{\frac{1}{2}}+h_j-h_{j_3})}} \left[ \ln |x_{12}|^2 \Phi_{j_3}(x_2, z_2) + \tilde{\Phi}_{j_3}(x_2, z_2) \right] \tag{55}$$

The only way to avoid having logarithms in both  $x$  and  $z$  in the WZNW model is therefore if all correlation functions of the primary field with  $j = 1/2$  vanish.

## 5 Conclusions

We have shown that the CFTs on the world-sheet and the boundary of  $AdS_3$  can both have logarithmic operators. We therefore have a duality between two LCFTs. In most cases, primary operators on the world sheet are mapped to primary operators on the boundary and logarithmic operators are mapped to logarithmic operators. However, because logarithmic operators on the boundary have to be in indecomposable representations of the global part of the Kac-Moody algebra on the world-sheet, but do not have to be in indecomposable representations of the world-sheet Virasoro algebra, it is possible for primary operators in one CFT to correspond to logarithmic operators in the other. The field with  $j = 1/2$  is one example – it is a logarithmic operator on the boundary but not on the world-sheet. An operator which was in an indecomposable of the world-sheet Virasoro algebra but not of the global  $SL(2)$  algebra would be logarithmic on the world-sheet but not on the boundary – the logarithmic operators which occur in finite dimensional representations of  $SL(2)$  are of this type. It is therefore possible that in other models a LCFT could be dual to an ordinary CFT – for example, there could be logarithmic operators in the world-sheet CFT for string theory on  $AdS_{D>3}$  which describe d-brane recoil in those theories, without there being logarithmic operators in the CFT on the boundary. We have also seen that most, but not all, of the logarithmic operators in either CFT correspond to dipole fields in the bulk.

The fields with  $j = 1/2$  play a crucial role in determining whether the WZNW model on  $SL(2, R)$  or  $SL(2, C)/SU(2)$  is a LCFT or not. This is the representation which has the minimum value of the conformal dimension in the discrete series, and the maximum value in the continuous series. It also corresponds to the minimum mass<sup>2</sup> for scalar fields in  $AdS_3$ , it is the only field which can be in an indecomposable representation of  $SL(2)$  but still be a primary field in an irreducible representation of the Virasoro algebra on the world sheet, and it is the only field which can couple to a logarithmic operator on the boundary of  $AdS_3$  even if it has the same equation of motion in the bulk as an ordinary scalar field. Like the puncture operator in minimal models, the operator  $\tilde{\Phi}_{\frac{1}{2}}$  is not a logarithmic operator (on the world-sheet) but logarithmic operators occur in the OPE of  $\tilde{\Phi}_{\frac{1}{2}}$  with other primary fields. However, unlike the puncture operator,  $\tilde{\Phi}_{\frac{1}{2}}$  is not in an irreducible representation of the complete symmetry algebra of the theory, and this provides a simple way to understand why the logarithmic operators must be included in a theory with  $\tilde{\Phi}_{\frac{1}{2}}$  in the spectrum.



The major question raised by these results is, what are the implications for string theory on  $AdS_3$ ? Logarithmic operators in string theory are thought to generate additional target space symmetries, and it would be interesting to know what zero modes in the string theory are generated by the logarithmic operators in this WZNW model— for instance, it has been suggested that logarithmic operators might restore the Poincare symmetry broken by the position of the branes in the D1-D5 system. It is therefore important to find out for which values of  $j$  the logarithmic operator as well as the primary operator is part of the spectrum. The simplest solution would be if the fields with  $j = 1/2$  decoupled completely from the spectrum, in which case there would be no logarithmic operators, but this seems to conflict with the 3-point functions calculated in [11, 12]. We have found that if there is a scalar field with  $m^2 = -1$  in the bulk theory, there must also be dipole fields with  $m^2 > -1$ . It would be interesting to understand why this should be true in the supergravity theory.

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